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INTERIOR REGULARITY ESTIMATES FOR ELLIPTIC SYSTEMS OF DIFFERENCE EQUATIONS

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ABSTRACT

We prove interior regularity estimates for a large class of difference approximations for elliptic systems of partial differential equations. These estimates are analogous to those for the systems of differential equations. From these regularity estimates we obtain estimates on the convergence of the solutions of the difference equations. A theory of pseudo-difference operators with order is developed and used to prove the regularity results. We also comment on factors to be considered in choosing a difference scheme for computations.

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1. Introduction

In this paper we present interior regularity estimates for finite difference schemes for elliptic systems of partial differential equations. We show that a large class of schemes for elliptic systems, which we call regular schemes, satisfy interior regularity estimates analogous to those of the corresponding differential equations. We also consider some non-regular schemes and show that they satisfy weaker regularity estimates.

By way of example, consider the Cauchy-Riemann equations

$$\begin{aligned} &\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \\ &\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \end{aligned} \tag{1.1}$$

on the unit square. One possible difference approximation to this system is a centered difference scheme such as

$$\begin{aligned} &\frac{u_{i+1,j} - u_{i-1,j}}{2h} - \frac{v_{i,j+1} - v_{i,j-1}}{2h} = 0 \\ &\frac{v_{i+1,j} - v_{i-1,j}}{2h} + \frac{u_{i,j+1} - u_{i,j-1}}{2h} = 0 \end{aligned} \tag{1.2}$$

$$i,j = 1,2,\cdots,N-1,$$

where $h = 1/N$ and u_{ij} and v_{ij} approximate $u(ih,jh)$ and $v(ih,jh)$, respectively. Another possible approach is to use a staggered mesh in which the variables u and v are defined at different points such as

$$\begin{aligned} &\frac{u_{i+1,j+\frac{1}{2}} - u_{i,j+\frac{1}{2}}}{h} - \frac{v_{i+\frac{1}{2},j+1} - v_{i+\frac{1}{2},j}}{h} = 0 \\ &\frac{v_{i+\frac{1}{2},j} - v_{i-\frac{1}{2},j}}{h} + \frac{u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}}}{h} = 0 \end{aligned} \tag{1.3}$$

where $u_{i,j+\frac{1}{2}}$ approximates $u(ih,(j+\frac{1}{2})h)$ and $v_{i+\frac{1}{2},j}$ approximates $v((i+\frac{1}{2})h,jh)$.

In section 5 we show that the solutions of the staggered scheme (1.3), which is a regular scheme, satisfy regularity estimates analogous to those of the Cauchy-Riemann equations themselves, while the solutions of the centered scheme (1.2), a non-regular scheme, satisfy weaker estimates. The regularity estimates for the staggered scheme (1.3) are a consequence of Theorem 2.1, the interior regularity estimate for general regular schemes. The failure of the solutions of the scheme (1.2) to satisfy regularity estimates analogous to those satisfied by solutions of the differential equations (1.1) is a direct consequence of the non-regularity of the scheme (1.2).

The outline of the paper is as follows. The main definitions and the statement of the main theorem are presented in section 2. The principal result of this paper is the interior regularity estimate, Theorem 2.1, which, for a regular elliptic system of difference equations $L u = f$, expresses the smoothness of the solution u in the interior of the domain Ω in terms of the smoothness of the data f in Ω. As an application we present Theorem 2.2 which states that if the solutions u_h of a regular elliptic system of difference equations converge as h tends to zero to the solution u of an elliptic system of differential equations, and if the difference equations are a sufficiently accurate approximation to the differential equations, then the difference quotients of $u_h - u$ converge to zero at the same rate that $u_h - u$ converges to zero. Section 3 is devoted to developing the theory of pseudo-difference operators which is used in the proofs of the regularity theorems. The interior regularity theorems are proved in section 4. In section 5 several examples of regular and non-regular difference schemes are presented. We also present some practical considerations which one might use in deciding on a numerical scheme for a particular problem.

We have been careful to separate the statements of the results from the methods used in the proofs. Therefore, the reader who is interested only in the results of this paper need not read sections 3 or 4 but rather can concentrate on section 5 where the method is applied to several examples with references to the main theorems of sections 2, 3, and 4.

Both regular and non-regular schemes have been used for computations with elliptic systems. For the Cauchy-Riemann equations a regular staggered scheme has been used by Ghil and Balgovind [1979] and Lomax and Martin [1974]. Frequently elliptic systems arise as part of a larger, more complex problem. The numerical solution of the Stokes equations or a similar elliptic system is often a part of an algorithm for the integration of the time-dependent incompressible Navier-Stokes equations. In this context staggered meshes have been used by Harlow and Welch [1964], Patankar and Spalding [1972], and others, while a centered scheme has been used by Chorin [1967, 1968]. For such computations the overall accuracy of the solution depends on many factors, such as boundary conditions, treatment of nonlinearities, etc., and thus the use of a non-regular scheme need not entail a less accurate solution. This topic is also discussed in section 5.

The theory of pseudo-difference operators developed in section 3 parallels the theory of pseudo-differential operators that has proven so useful for the study of partial differential equations. We refer the reader to Nirenberg [1970] and Taylor [1974] for introductions to the theory and applications of pseudo-differential operators. Pseudo-difference and pseudo-translation operators have been studied previously by Vaillancourt [1969], Lax and Nirenberg [1966], and Yamaguti and Nogi [1967]. The theory presented here differs from these other theories in that it emphasizes the order of a pseudo-difference operator. Considering pseudo-difference operators with order allows one to obtain the regularity estimates in a way that is analogous to that used in the theory of pseudo-differential operators.

The interior regularity estimates given in sections 2 and 4 are analogous to Sobolev interior regularity estimates for elliptic systems of partial differential equations; see Agmon et al. [1964, p. 77]. Interior regularity estimates for difference approximations to a single elliptic equation have been given by Thomée and Westergren [1968]. Similar results have been given by Vainikko and Tamme [1976].

Brandt and Dinar [1979] have discussed regularity for systems of elliptic difference schemes. Our definition of a regular difference scheme corresponds to their definition of T-ellipticity. They note that schemes which lack regularity (i.e. are not T-elliptic) can cause numerical solution procedures, such as the multi-level adaptive technique, to be unstable unless great care is taken.

Using the interior regularity estimate we show how to obtain estimates on the convergence of higher order differences of the solutions to the difference equations to the corresponding derivatives of the solution of the differential equation. These results have been obtained by Thomée and Westergren [1968] and Vainikko and Tamme [1976] for a single elliptic equation. Bramble and Hubbard [1964] proved similar convergence results for a single elliptic equation by using the discrete Green's function.

For the finite element method, Fix et al. [1977] emphasize the importance of regularity estimates in obtaining optimal rates of convergence as the mesh is refined. The Grid Decomposition Property of Fix et al. [1977] is the requirement that a regularity estimate holds for the discrete first-order Poisson equation with a particular form of the data. It appears to be nontrivial to show whether or not a given finite element space satisfies the Grid Decomposition Property. In contrast, to check the regularity of a finite difference scheme involves only algebraic manipulation. The regularity estimate for the difference equations then follows from the regularity of the scheme. For the finite element method applied to a single elliptic equation, interior convergence estimates have been obtained by Bramble and Thomée [1974].

In a forthcoming paper, we will prove regularity estimates up to the boundary in domains with boundary-fitted coordinate systems. We will use these regularity estimates to derive estimates on the rate of convergence of the solutions of the difference approximations to the solution of the system of differential equations.

2. Preliminaries and Statement of the Main Theorem

We define elliptic systems of partial differential equations as was done by Douglis and Nirenberg [1955].

<u>Definition 2.1</u> For $i,j = 1,\cdots,n$, let $\ell_{ij}(x,D)$ be a linear differential operator expressed as a polynomial in $D = (-i\frac{\partial}{\partial x_1},\cdots,-i\frac{\partial}{\partial x_d})$ with variable coefficients depending on $x \in \mathbb{R}^d$. The system of partial differential equations

$$(2.1)\qquad \sum_{j=1}^{n} \ell_{ij}(x,D)u_j = f_i(x) \qquad i=1,\cdots,n$$

is elliptic if there are sets of integers $\{\sigma_i\}_{i=1}^n$ and $\{\tau_j\}_{j=1}^n$ such that the polynomials $\ell_{ij}(x,\xi)$ as polynomials in $\xi \in \mathbb{R}^d$ satisfy

$$\deg\ \ell_{ij}(x,\xi) \leq \sigma_i + \tau_j \qquad i,j=1,\cdots,n$$

and

$$\det\ \ell'_{ij}(x,\xi) \neq 0 \qquad \text{for}\ \ \xi \neq 0$$

where $\ell'_{ij}(x,\xi)$ is that part of $\ell_{ij}(x,\xi)$ that is of degree $\sigma_i + \tau_j$ in ξ. A polynomial of negative degree is identically zero by definition.

Note that $\det\ \ell'_{ij}(x,\xi)$ is a polynomial in ξ which is homogeneous of degree $2p = \sum_{i=1}^{n} (\sigma_i + \tau_i)$. The determinant condition in the definition is equivalent to

$$(2.2) \qquad |\det\ \ell_{ij}(x,\xi)| \geq C|\xi|^{2p} \qquad \text{for}\ \ |\xi|\ \ \text{large}$$

where $|\xi|^2 = \sum_{i=1}^{d} \xi_i^2$. Without loss of generality, we may assume that $\max\ \sigma_i = 0$ and $\min\ \tau_j \geq 0$.

Several examples of elliptic systems with appropriate values for the σ_i's and τ_j's are:

1) The inhomogeneous Cauchy-Riemann equations

$$(2.3) \qquad \begin{aligned} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= f_1(x,y) \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= f_2(x,y) \end{aligned}$$

$$\sigma_1 = \sigma_2 = 0 \qquad \tau_1 = \tau_2 = 1.$$

2) The first-order Poisson equations

$$(2.4) \qquad \begin{aligned} u - \frac{\partial p}{\partial x} &= f_1(x,y) \\ v - \frac{\partial p}{\partial y} &= f_2(x,y) \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= f_3(x,y) \end{aligned}$$

$$\sigma_1 = \sigma_2 = -1, \quad \sigma_3 = 0, \quad \tau_1 = \tau_2 = 1, \quad \tau_3 = 2.$$

3) The Stokes equations

$$(2.5) \qquad \begin{aligned} \nabla^2 u + \frac{\partial p}{\partial x} &= f_1(x,y) \\ \nabla^2 v + \frac{\partial p}{\partial y} &= f_2(x,y) \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= f_3(x,y) \end{aligned}$$

$$\sigma_1 = \sigma_2 = 0, \quad \sigma_3 = -1, \quad \tau_1 = \tau_2 = 2, \quad \tau_3 = 1.$$

Before defining regular elliptic systems of difference equations, we must introduce some notation. For $h > 0$ and $\mu \in \mathbb{Z}^d$, the translation operator T_h^μ is defined by

$$(T_h^\mu u)(x) = u(x + h\mu) \qquad \text{for } x \in \mathbb{R}^d .$$

The forward and backward difference operators $\delta_+^{e_j}$ and $\delta_-^{e_j}$ are given by

$$\delta_+^{e_j} = \frac{1}{h}(T_h^{e_j} - I)$$

$$\delta_-^{e_j} = \frac{1}{h}(I - T_h^{-e_j})$$

where $\{e_j\}_{j=1}^{d}$ is the usual basis for $\mathbb{R}^d$. For $\mu,\nu \in \mathbb{Z}^d$, we write $\mu \le \nu$ if $\mu_j \le \nu_j$ for $j=1,\cdots,d$. If $\alpha \in \mathbb{Z}^d$ and $\alpha \ge 0$, we say that α is a multi-index and we define

$$|\alpha| = \sum_{j=1}^{d} \alpha_j \quad .$$

If α is a multi-index, define

$$\delta_{\pm}^{\alpha} = \prod_{j=1}^{d} (\delta_{\pm}^{e_j})^{\alpha_j} \ .$$

A translation operator is an operator A of the form

$$Au = \sum_{\mu \in M} a_\mu(h,x) T_h^\mu u \quad , \tag{2.6}$$

where M is a finite subset of $\mathbb{Z}^d$ and each a_μ is a smooth function of $x \in \mathbb{R}^d$ for each $h > 0$. For convenience, we will always assume $0 \in M$ even if $a_0(h,x) \equiv 0$. We assume here and subsequently that $h \le h_{max}$ for some fixed h_{max}.

The symbol of the translation operator A given by equation (2.6) is the function

$$a(h,x,\zeta) = \sum_{\mu \in M} a_\mu(h,x) e^{i\mu\cdot\zeta}. \tag{2.7}$$

The symbol $a(h,x,\zeta)$ will usually be evaluated at $\zeta = h\xi$ for $\xi \in \mathbb{Z}^d$. $a(h,x,\zeta)$ is 2π-periodic in each ζ_j. Also

$$A(e^{ix\cdot\xi}) = a(h,x,h\xi)e^{ix\cdot\xi}.$$

We only allow translates which are integer multiples of the mesh width h. This is to keep track of the smallest effective distance between grid points. This does not eliminate staggered grids: we view fractional translates as a change of the dependent variables instead of part of the translation operator. For example, if u and v are defined on the grids $(ih, (j+\frac{1}{2})h)$ and $((i+\frac{1}{2})h, jh)$ in $\mathbb{R}^2$, as in the example in section 1, then the closest points in the union of the two grids are distance $h/\sqrt{2}$ apart; but u and v are both defined on uniform grids of mesh width h. The regularity properties of u and v should be examined on grids of mesh width h, not $h/2$ or $h/\sqrt{2}$. As in computer programming where integer subscripts must be used, we can define new variables

$$\tilde{u}_{ij} = u_{i,j+\frac{1}{2}}, \qquad \tilde{v}_{ij} = v_{i+\frac{1}{2},j},$$

and write the translation operator in terms of $\tilde{u}$ and $\tilde{v}$.

The translation operator (2.6) is called a difference operator of order m, where m is a non-negative integer, if it can be written as a sum of terms of the form

$$a_{t,\alpha}(h,x)\; h^t\; \delta_+^\alpha\; T_h^{\nu-}, \tag{2.8}$$

where $\nu_j^- = \min\limits_{\mu \in M} \mu_j$ and each such term satisfies:

(i) $\quad t \geq 0, \quad \alpha \geq 0$

(ii) $\quad |\alpha| - t \leq m$

(iii) $a_{t,\alpha}$ is a smooth function of $x \in \mathbb{R}^d$ for each $h > 0$

(iv) $a_{t,\alpha}$ and sufficiently many of its x derivatives are bounded, independent of x and h.

For example, the centered difference operator in $\mathbb{R}^1$ is a difference operator of order 1,

$$\frac{1}{2h}(T^1 - T^{-1}) = (\delta_+^1 + \frac{h}{2}\,\delta_+^2)T^{-1} \quad .$$

If $a(x)$ is smooth, $Au = \delta_+\delta_-(a(x)u)$ is a difference operator of order 2,

$$A = \Big((\delta_+\delta_- a(x))\,I + 2\big(\delta_+ a(x)\big)\delta_+ + \big(T^1 a(x)\big)\,\delta_+^2\Big)T^{-1}.$$

To write a translation operator (2.6) as a sum of terms of the form (2.8), factor out $T_h^{\nu-}$ from the right of A, use $T_h^{e_j} = I + h\delta_+^{e_j}$ to replace all remaining occurences of T by I and δ_+, and group terms by powers of δ_+.

Not every translation operator is a difference operator, e.g. $h^{-1}T_h^{e_1}$, but all standard difference schemes are in the class of difference operators. We use this definition of difference scheme because it has a well-defined concept of order and yet is very general. Also it will enable us to show easily that difference operators are pseudo-difference operators of the same order as defined in section 3.

We need a symbol which plays the same role for difference operators that $|\xi|^2$ plays for differential operators. For $\zeta \in \mathbb{R}^d$, let

$$\Lambda_j(h,\zeta) = \frac{2}{h}\left|\sin\frac{\zeta_j}{2}\right| = \frac{|e^{i\zeta_j}-1|}{h} \qquad j = 1,\cdots,d$$

(2.9)
$$\Lambda_0(h,\zeta)^2 = \sum_{j=1}^{d} \Lambda_j(h,\zeta)^2$$

$$\Lambda(h,\zeta)^2 = 1 + \Lambda_0(h,\zeta)^2$$

$\Lambda(h,\zeta)$ is even and 2π-periodic in each ζ_j. Note that $\Lambda_j(h,\zeta)^2$ is the symbol of $-\delta_+^{e_j}\delta_-^{e_j}$.

<u>Definition 2.2</u> For $i,j = 1,\cdots,n$, let L_{ij} be a difference operator with symbol $\ell_{ij}(h,x,\zeta)$. The system of difference equations

(2.10)
$$\sum_{j=1}^{n} L_{ij}\, u_j(x) = f_i(x) \qquad i=1,\cdots,n$$

is a regular elliptic system if there are sets of integers $\{\sigma_i\}_{i=1}^n$ and $\{\tau_j\}_{j=1}^n$ such that each L_{ij} is a difference operator of order at most $\sigma_i + \tau_j$, and if there are positive constants C, ξ_0, h_0 such that

(2.11)
$$|\det\ \ell_{ij}(h,x,\zeta)| \geq C\,\Lambda(h,\zeta)^{2p}$$

for $h\xi_0 \leq |\zeta|_\infty \leq \pi$ and $0 < h \leq h_0$ where $2p = \sum_{i=1}^{n}(\sigma_i+\tau_i)$. A difference operator of negative order is identically zero by definition. We will say the system (2.10) is regular elliptic of order (σ,τ).

Note that the scheme (1.3) is regular elliptic while the scheme (1.2) is not; see section 5. Without loss of generality, we may assume that $\max \sigma_i = 0$ and $\min \tau_j \geq 0$.

This definition is equivalent to the definition of T-ellipticity given by Brandt and Dinar [1979]. The definition of an elliptic difference equation given by Thomée and Westergren [1968] requires the scheme to be a consistent approximation of an elliptic differential equation and have no lower order terms; if this is so, their definition is equivalent to Definition 2.2.

We now consider grids in a bounded domain $\Omega \subset \mathbb{R}^d$. For each $h > 0$ let $G(h,\Omega)$ be a grid in Ω. We assume that at each point $x_0 \in \Omega$ there is a neighborhood $N(x_0)$ and a smooth map independent of h mapping $N(x_0)$ into $\mathbb{R}^d$ such that the image of $N(x_0) \cap G(h,\Omega)$ is a uniform rectangular grid of mesh width h parallel to the coordinate axes. For simplicity, we will treat $G(h,\Omega)$ as if it were a uniform rectangular grid on Ω itself.

We define difference norms in subdomains Ω_1 of Ω. Unlike derivatives, the value of $\delta_+^\alpha u$ at a point x does not depend only on values of u in arbitrarily small neighborhoods of x. For any $x \in \mathbb{R}^d$ and multi-index α, define the d-dimensional rectangle

$$R(h,x,\alpha) = \{y \in \mathbb{R}^d : \; x_j \le y_j \le x_j + h\alpha_j, \quad j = 1,\cdots,d\}. \tag{2.12}$$

For a given h and x, $\delta_+^\alpha u(x)$ depends on values of u in the rectangle $R(h, x,\alpha)$. We will include in the difference norm of u over Ω_1 any difference whose value depends on values of u in Ω_1. Thus we define

$$G(h,\Omega_1,\alpha) = \{x \in G(h,\Omega) : \; R(h,x,\alpha) \cap \Omega_1 \neq \emptyset \text{ and } R(h,x,\alpha) \subset \Omega\} .$$

For any integer $s \geq 0$ and subdomain $\Omega_1 \subset \Omega$, define the difference norm

$$\|u\|^2_{h,s,\Omega_1} = \sum_{0 \leq |\alpha| \leq s} h^d \sum_{x \in G(h,\Omega_1,\alpha)} |(\delta_+^\alpha u)(x)|^2 \quad . \tag{2.13}$$

These norms are a discrete version of the integral norms

$$\|u\|^2_{s,\Omega_1} = \sum_{0 \leq |\alpha| \leq s} \int_{\Omega_1} |D^\alpha u|^2 \, dx \quad . \tag{2.14}$$

If τ is a multi-index, define

$$\|u\|^2_{h,s+\tau,\Omega_1} = \sum_{j=1}^{n} \|u_j\|^2_{h,\Omega_1,s+\tau_j} \quad . \tag{2.15}$$

We say that a translation operator A given by equation (2.6) is defined in Ω if, for each $h > 0$, each $a_\mu(h,x)$ is defined and is smooth for x in

$$\Omega_{h,M} = \{x \in \Omega\colon \ R(h, x + h\nu^-, \nu^+ - \nu^-) \subset \Omega\} \quad , \tag{2.16}$$

where $\nu^+, \nu^- \in \mathbb{Z}^d$ are given by

$$\nu_j^+ = \max_{\mu \in M} \mu_j \qquad \nu_j^- = \min_{\mu \in M} \mu_j \quad .$$

Since we assume $0 \in M$, $\nu^- \leq 0 \leq \nu^+$, so $\Omega_{h',M} \subset \Omega_{h,M}$ if $h \leq h'$. Also $\bigcup_{h>0} \Omega_{h,M} = \Omega$. We say that a translation operator defined in Ω is a difference operator defined in Ω if it satisfies the conditions for a difference operator for $x \in \Omega_{h,M}$. Note that for any α which occurs in any term of the form (2.8) when A is expressed as a sum of such terms, $\alpha \leq \nu^+ - \nu^-$, so $R(h, x+h\nu^-, \alpha) \subset R(h, x+h\nu^-, \nu^+ - \nu^-)$. So $G(h,\Omega) \cap \Omega_{h,M}$ is the most natural subset of the grid $G(h,\Omega)$ in which $Au(x)$ is defined for a given grid function u on $G(h,\Omega)$, whether we write A in translation form (2.6) or in difference form as a sum of terms of the form (2.8). If, for a fixed h, u and f are functions on $G(h,\Omega)$, then $Au = f$ on $G(h,\Omega)$ means $Au(x) = f(x)$ for $x \in G(h,\Omega) \cap \Omega_{h,M}$.

We now state the interior regularity estimate for regular elliptic systems of difference equations.

<u>Theorem 2.1</u> If system (2.10) is a regular elliptic system of difference equations of order (σ,τ) defined in Ω and u is any solution of the system on the grid $G(h,\Omega)$, then for any integer $k \geq 0$ and subdomain Ω_1 with compact closure $\overline{\Omega}_1 \subset \Omega$,

$$(2.17) \qquad \| u \|_{h,k+\tau,\Omega_1} \leq C(\| f \|_{h,k-\sigma,\Omega} + \| u \|_{h,0,\Omega}) \ .$$

The constant C depends on k, Ω_1, and Ω, but is independent of u and h.

The proof of this theorem is in section 4.

Theorem 2.1 is analogous to the interior regularity theorem for elliptic systems of partial differential equations given by Agmon, et al. [1964, p. 77], which we restate here for completeness.

<u>Theorem 2.1'</u> If $u(x)$ is any solution of the elliptic system of partial differential equations (2.1) in a domain $\Omega \subset \mathbb{R}^d$, then for any integer $k \geq 0$ and subdomain Ω_1 with compact closure $\overline{\Omega}_1 \subset \Omega$,

$$\| u \|_{k+\tau,\Omega_1} \leq C(\| f \|_{k-\sigma,\Omega} + \| u \|_{0,\Omega}) \ .$$

The constant C depends on k, Ω_1, and Ω, but is independent of u.

One consequence of the interior regularity estimate (2.17) is a convergence theorem proved by Thomée and Westergren [1968] for single elliptic equations. Their result extends to regular elliptic systems of difference equations.

<u>Theorem 2.2</u> Let L_h be a regular elliptic difference operator of order (σ,τ) defined in Ω, let u_h and f_h be functions on the grid $G(h,\Omega)$ with $L_h u_h = f_h$, and let u be a function of x in Ω. Suppose that

$$\| u_h - u \|_{h,0,\Omega} = O(h^r) \tag{2.18}$$

and

$$\| f_h - L_h u \|_{h,k-\sigma,\Omega} = O(h^s) \tag{2.19}$$

for some positive integer k as $h \to 0$; then

$$\| u_h - u \|_{h,k+\tau,\Omega_1} = O(h^t) \tag{2.20}$$

as $h \to 0$ where Ω_1 is any subdomain with compact closure $\overline{\Omega}_1 \leq \Omega$ and $t = \min(r,s)$.

<u>Proof</u> Equation (2.20) is a direct consequence of estimate (2.17) applied to $u_h - u$ followed by the application of the estimates (2.18) and (2.19).

Equation (2.19) can be obtained if $L_h u_h = f_h$ approximates with order of accuracy s an elliptic system of difference equations $Lu = f$ of which u is a solution. If u is smooth, then its difference quotients $\delta_+^\alpha u$ converge to its derivatives $\partial^\alpha u$, so results can be obtained on the rate of convergence of the difference quotients of u_h to the derivatives of u. Similarly, if Q_h is a difference operator approximating a differential operator Q with order of accuracy q then $Q_h u_h$ converges to Qu at a rate depending on t and q. Supremum-norm convergence results can be obtained from ℓ^2 convergence results using discrete Sobolev inequalities. The reader is referred to Thomée and Westergren [1968] for details.

3. Pseudo-Difference Operators with Order

We develop here the theory of pseudo-difference operators with order. It suffices to use the torus $T^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$, i.e., $[0,2\pi]^d$ with opposite faces identified, as our basic domain. Let $N > 0$ be an integer, and $h = \frac{2\pi}{2N+1}$. For $\nu = (\nu_1,\cdots,\nu_d)$ where the ν_j's are integers, let $x_\nu = h\nu$. Define the grid

$$G = \{x_\nu:\ 0 \le \nu_j \le 2N \quad \text{for} \quad j=1,\cdots,d\} .$$

We consider $\mathbb{C}^n$-valued functions $u(x)$ defined for $x \in G$. Define an inner product on this set of functions by

$$(u,v)_G = \hbar^d \sum_{x \in G} u(x)^* v(x) \quad ,$$

where $\hbar = \frac{h}{2\pi} = \frac{1}{2N+1}$.

The discrete Fourier transform of u is

$$\hat{u}(\xi) = \hbar^d \sum_{x \in G} e^{-ix\cdot\xi} u(x) = (e^{ix\cdot\xi}, u)_G \qquad (\xi \in \mathbb{Z}^d) .$$

$\hat{u}(\xi)$ is $(2N+1)$-periodic in each ξ_j. Let

$$\Gamma = \{\xi \in \mathbb{Z}^d : \ |\xi_j| \leq N \ \text{ for } \ j=1,\cdots,d\} .$$

The set of functions $e^{ix\cdot\xi}e_j$ for $\xi \in \Gamma$ and $j=1,\cdots,n$ (where $\{e_j\}$ is the standard basis of $\mathbb{C}^n$) form an orthonormal basis of $\ell^2(G)^n$ with the inner product $(\cdot,\cdot)_G$. The Fourier inversion formula is

$$u(x) = \sum_{\xi \in \Gamma} e^{ix\cdot\xi}\, \hat{u}(\xi) , \qquad (x \in G)$$

and Parseval's formula is

$$(u,v)_G = \sum_{\xi \in \Gamma} \hat{u}(\xi)^* \hat{v}(\xi) .$$

The translation operators T_h^μ and forward difference operators δ^α (dropping the +) defined in section 2 satisfy

$$\widehat{T_h^\mu u}(\xi) = e^{ih\mu\cdot\xi}\, \hat{u}(\xi)$$

$$\widehat{\delta^\alpha u}(\xi) = \frac{1}{h^{|\alpha|}}\left(e^{ih\xi_1} - 1\right)^{\alpha_1} \cdots \left(e^{ih\xi_d} - 1\right)^{\alpha_d} \hat{u}(\xi) .$$

The norms we use are discrete Sobolev norms. The quantity $\Lambda(h,h\xi)^2$, where $\Lambda(h,\zeta)$ is defined by equations (2.9), corresponds to the $1 + |\xi|^2$ factor in the definition of continuous Sobolev norms. Therefore, if we set $\lambda_j(\xi) = \Lambda_j(h,h\xi)$ and $\lambda(\xi) = \Lambda(h,h\xi)$, then for $s \in \mathbb{R}$ the discrete Sobolev norm is defined by

$$\|u\|_s^2 = \sum_{\xi \in \Gamma} \lambda(\xi)^{2s} |\hat{u}(\xi)|^2 .$$

Note that $\| u \|_0$ is the $\ell^2(G)^n$ norm of u:

$$\| u \|_0^2 = (u,u)_G .$$

Since $\left|\frac{1}{h}(e^{ih\xi_j} - 1)\right| = \lambda_j(\xi)$, Parseval's formula gives

$$\| \delta^\alpha u\|_0^2 = \sum_{\xi \in \Gamma} \left(\lambda_1(\xi)^{\alpha_1} \cdots \lambda_d(\xi)^{\alpha_d}\right)^2 |\hat{u}(\xi)|^2 .$$

If s is a non-negative integer, then $\| u \|_s$ is an equivalent norm to

$$\| u \|_{h,s,T^d} = \left(\sum_{0 \le |\alpha| \le s} h^d \sum_{x \in G} |(\delta^\alpha u)(x)|^2 \right)^{\frac{1}{2}} = \left((2\pi)^d \sum_{|\alpha| \le s} \| \delta^\alpha u \|_0^2 \right)^{\frac{1}{2}}$$

and the constants of the equivalence are independent of h. Notice that h and T^d do not appear explicitly in the notation for the discrete Sobolev norms. We trust no confusion will arise.

We will need the following analog of Peetre's inequality.

<u>Lemma 3.1</u> If $s \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^d$, then $\lambda(\xi-\eta)^{2s} \le 2^{|s|} \lambda(\xi)^{2|s|} \lambda(\eta)^{2s}$.

<u>Proof</u> If $s = 1$, this follows from the inequality

$$\sin^2(\theta_1 - \theta_2) \le 2(\sin^2 \theta_1 + \sin^2 \theta_2)$$

and the $s \ge 0$ case follows by taking powers.

If $s<0$, replace η by $\xi-\eta$ in the $s = 1$ case and raise both sides to the $|s|$ power.

In what follows, the dependence of G, Γ, $\lambda(\xi)$, N, etc. on h is understood.

Definition 3.1 Let $m \in \mathbb{R}$. If $p(h,x,\zeta)$ is an $n \times n$ matrix function of h, $x \in G$, and $\zeta \in \mathbb{R}^d$, 2π-periodic in each ζ_j, we say that $p(h,x,\zeta) \in S^m_{J,K}$ if there is a constant C such that

$$|\partial^\beta_\zeta \, \delta^\alpha_x \, p(h,x,\zeta)| \le C\, h^{-|\beta|}\, \Lambda(h,\zeta)^{m-|\beta|},$$

for all h, x,ζ and for $|\alpha| \le J$, $|\beta| \le K$. We will write $p(h,x,\zeta) \in S^m$ because we will always assume that J and K are large enough to make the following theory valid. We have used the notation $\partial^\beta_\zeta = \left(\frac{\partial}{\partial \zeta_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial \zeta_d}\right)^{\beta_d}$, and δ^α_x operates on $p(h,x,\zeta)$ as a function of x. $p(h,x,\zeta)$ is called a symbol of order m. If $p(h,x,\zeta) \in S^m_{J,K}$ for all $m \in \mathbb{R}$, we will write $p(h,x,\zeta) \in S^{-\infty}_{J,K}$.

Definition 3.2 If $p(h, x, \zeta) \in S^m$, define the operator P on $\ell^2(G)^n$ by

$$(3.1) \qquad (Pu)\,(x) = \sum_{\xi \in \Gamma} e^{ix\cdot\xi}\, p(h, x, h\xi)\, \hat{u}(\xi).$$

P is called a pseudo-difference operator of order m and we write $P \in PS^m$. The symbol of P, $p(h, x, \zeta)$, is sometimes denoted by σ_P.

Another useful representation of the operator P gives the Fourier transform of Pu in terms of the Fourier transform of u and the Fourier transform in x of the symbol of P which is

$$\hat{p}(h, \eta, \zeta) = \not{h}^d \sum_{x \in G} e^{-ix\cdot\eta}\, p(h, x, \zeta).$$

This representation is:

$$\widehat{Pu}(\eta) = h^d \sum_{x \in G} e^{-ix\cdot\eta} \sum_{\xi \in \Gamma} e^{ix\cdot\xi} \; p(h, x, h\xi)\; \hat{u}(\xi)$$

$$= \sum_{\xi \in \Gamma} \left(h^d \sum_{x \in G} e^{-ix\cdot(\eta-\xi)} \; p(h, x, h\xi)\right) \hat{u}(\xi)$$

or

$$\widehat{Pu}\;(\eta) = \sum_{\xi \in \Gamma} \hat{p}(h, \eta-\xi, h\xi)\; \hat{u}(\xi). \tag{3.2}$$

Clearly, if $p(h, x, \zeta) \in S^m_{J,K}$, then $\delta^\alpha_x\, p(h, x, \zeta) \in S^m_{J-|\alpha|,K}$ and $h^{|\beta|}\, \partial^\beta_\zeta\; p(h, x, \zeta) \in S^{m-|\beta|}_{J,K-|\beta|}$. Since $h\Lambda(h,\zeta)$ is bounded, $h^t\, p(h, x, \zeta) \in S^{m-t}_{J,K}$ if $t \geq 0$. If $q(h, x, \zeta) \in S^{m_1}_{J,K}$ and $r(h,x,\zeta) \in S^{m_2}_{J,K}$, then $q(h, x, \zeta)\, r(h, x, \zeta) \in S^{m_1+m_2}_{J,K}$.

Every difference operator of order m defined on the torus T^d is a pseudo-difference operator of order m. This is because the symbol of T^μ_h, $e^{i\mu\cdot\zeta}$, is in $S^0_{J,K}$ and the symbol of δ^{e_j}, $\frac{1}{h}(e^{i\zeta_j} - 1)$, is in $S^1_{J,K}$ for all J,K. So the symbol of $\delta^\alpha T^\mu_h$ is in $S^{|\alpha|}_{J,K}$ for all J,K. Since

$$|\delta^\alpha a(x)| \leq \max_{y \in R(h,x,\alpha)} |D^\alpha a(y)|$$

for any smooth function (where $R(h,x,\alpha)$ is the d-dimensional rectangle given by (2.12)), the remarks above imply that the symbols of operators of the form (2.8) which satisfy the conditions listed after (2.8) are in S^m.

It is also easy to check that for each $m \in \mathbb{R}$, $\Lambda(h,\zeta)^m \in S^m_{J,K}$ for all

J,K. Induction on $|\beta|$ shows that $\partial_\zeta^\beta \Lambda(h,\zeta)^m$ is a linear combination of terms of the form

$$h^{-a} \Lambda(h,\zeta)^{m-a} \prod_{j=1}^{d} \left(\sin \frac{\zeta_j}{2}\right)^{b_j} \left(\cos \frac{\zeta_j}{2}\right)^{c_j}$$

where a, b_j, c_j are non-negative integers satisfying

$$a \leq \sum_{j=1}^{d} b_j + |\beta| .$$

The desired result follows immediately since $h\Lambda(h,\zeta)$ and $h^{-1} \Lambda(h,\zeta)^{-1} |\sin \frac{\zeta_j}{2}|$ are both bounded. We write Λ^m for the operator with symbol $\Lambda(h,\zeta)^m$.

The order of a pseudo-difference operator determines the degree of smoothness (measured in terms of the discrete Sobolev norms) which is lost or gained when P is applied to a grid function u.

Theorem 3.1 If $p(h,x,\zeta) \in S^m$, then there is a constant C independent of h and u such that

$$\|Pu\|_{s-m} \leq C\|u\|_s .$$

We first need a lemma.

Lemma 3.2 If $p(h,x,\zeta) \in S^m_{J+r,K}$ with $r > d$, then

$$(3.3) \quad \sup_{|\beta| \leq K} \sum_{\eta \in \Gamma} \lambda(\eta)^J \sup_\zeta \left|\partial_\zeta^\beta \hat{p}(h,\eta,\zeta) h^{|\beta|} \Lambda(h,\zeta)^{-m+|\beta|}\right| \leq C$$

where C is independent of h. Conversely, if $p(h, x, \zeta)$ is a function of h, $x \in G$, and $\zeta \in \mathbb{R}^d$, 2π-periodic in each ζ_j, for which (3.3) holds, then $p(h,x,\zeta) \in S^m_{J,K}$.

If $p(h,x,\zeta)$ satisfies (3.3), we say $p(h,x,\zeta) \in \tilde{S}^m_{J,K}$.

<u>Proof</u> For $|\alpha| \leq J + r$, $|\beta| \leq K$,

$$|\lambda_1(\eta)^{\alpha_1} \cdots \lambda_d(\eta)^{\alpha_d} \partial^\beta_\zeta \hat{p}(h,\eta,\zeta)| = |\widehat{\delta^\alpha_x \partial^\beta_\zeta p}(h,\eta,\zeta)|$$

$$\leq \sup_{x \in G} \left|\delta^\alpha_x \partial^\beta_\zeta p(h,x,\zeta)\right| \leq C h^{-|\beta|} \Lambda(h,\zeta)^{m-|\beta|}.$$

Summing over $|\alpha| \leq J+r$ with appropriate weights,

$$\lambda(\eta)^{J+r} |\partial^\beta_\zeta \hat{p}(h,\eta,\zeta) h^{|\beta|} \Lambda(h,\zeta)^{-m+|\beta|}| \leq C.$$

The result follows since

$$\sum_{\eta \in \Gamma} \lambda(\eta)^{-r} < \sum_{\eta \in \mathbb{Z}^d} (1+|\eta|)^{-r} < \infty .$$

Conversely, if (3.3) holds, then

$$|\partial^\beta_\zeta \delta^\alpha_x p(h,x,\zeta)| \leq \sum_{\eta \in \Gamma} \lambda(\eta)^{|\alpha|} |\partial^\beta_\zeta \hat{p}(h,\eta,\zeta)|$$

so if $|\alpha| \leq J$, $|\beta| \leq K$, then

$$\left|\partial_\zeta^\beta\ \delta_x^\alpha\ p(h,x,\zeta)\ h^{|\beta|}\ \Lambda(h,\zeta)^{-m+|\beta|}\right| \ \leq\ C\ .$$

<u>Proof of Theorem 3.1</u> Let $\hat{v}(\eta) = \lambda(\eta)^{2(s-m)}\ \widehat{Pu}(\eta)$. Then

$$\|\ Pu\ \|_{s-m}^2 = \sum_{\eta\in\Gamma} \hat{v}(\eta)^*\ \widehat{Pu}(\eta)$$

$$= \sum_{\eta,\xi\in\Gamma} \hat{v}(\eta)^*\ \hat{p}(h,\eta-\xi,\ h\xi)\ \hat{u}(\xi)$$

$$= \sum_{\eta,\xi\in\Gamma} \hat{v}(\eta+\xi)^*\ \hat{p}(h,\eta,h\xi)\ \hat{u}(\xi)$$

by the (2N+1)-periodicity of $\hat{v}(\eta)$ and $\hat{p}(h,\eta,h\xi)$ in each η_j.

Applying the Schwarz inequality and Lemmas 3.1 and 3.2,

$$\|\ Pu\ \|_{s-m}^2 \leq \sum_{\eta,\xi\in\Gamma} \left(\sup_\xi \left|\hat{p}(h,\eta,h\xi)\lambda(\xi)^{-m}\right|\right)\left(\lambda(\xi)^m\lambda(\eta+\xi)^{s-m}\left|\hat{u}(\xi)\right|\right)\left(\lambda(\eta+\xi)^{s-m}\left|\widehat{Pu}(\eta+\xi)\right|\right)$$

$$\leq \sum_{\eta\in\Gamma} \sup_\xi \left|\hat{p}\ (h,\eta,h\xi)\lambda(\xi)^{-m}\right| \left[\sum_{\xi\in\Gamma} \lambda(\xi)^{2m}\ \lambda(\eta+\xi)^{2(s-m)}\ \left|\hat{u}(\xi)\right|^2\right]^{\frac{1}{2}} \|Pu\|_{s-m}$$

$$\leq\ 2^{|s-m|/2} \sum_{\eta\in\Gamma} \lambda(\eta)^{|s-m|}\ \sup_\xi \left|\hat{p}(h,\eta,h\xi)\lambda(\xi)^{-m}\right| \cdot \|u\|_s \cdot \|\ Pu\|_{s-m}$$

$$\leq\ C\ \|u\|_s\ \cdot\ \|\ Pu\|_{s-m}\ .$$

Divide by $\|\ Pu\|_{s-m}$ to obtain the theorem.

The product of two pseudo-difference operators is again a pseudo-difference operator whose symbol is, to highest order, the product of the symbols of the two operators.

<u>Theorem 3.2</u> If $q(h,x,\zeta) \in S^{m_1}$, $r(h,x,\zeta) \in S^{m_2}$, and $P=QR$, then P is a pseudo-difference operator with symbol $p(h,x,\zeta) \in S^{m_1+m_2}$ given by

$$\hat{p}(h,\eta,\zeta) = \sum_{\theta \in \Gamma} \hat{q}(h, \eta-\theta,\zeta+ h\theta)\ \hat{r}\ (h,\theta,\zeta). \tag{3.4}$$

Moreover, if $p_1(h,x,\zeta) = p(h,x,\zeta) - q(h,x,\zeta)\ r(h,x,\zeta)$,

then $p_1(h,x,\zeta) \in S^{m_1+m_2-1}$.

<u>Proof</u> $\widehat{Pu}\ (\eta) = \sum_{\theta \in \Gamma} \hat{q}(h,\eta-\theta,h\theta)\ \widehat{Ru}\ (\theta)$

$$= \sum_{\theta,\xi \in \Gamma} \hat{q}(h, \eta-\theta, h\theta)\ \hat{r}(h, \theta-\xi, h\xi)\ \hat{u}(\xi)$$

$$= \sum_{\xi \in \Gamma} \left[\sum_{\theta \in \Gamma} \hat{q}(h, \eta-\xi-\theta, h\xi + h\theta)\ \hat{r}(h, \theta, h\xi)\right] \hat{u}(\xi)$$

by the $(2N+1)$ - periodicity of $\hat{q}(h,\eta,\zeta)$ and $\hat{r}(h,\eta,\zeta)$ in each η_j and the 2π-periodicity of $\hat{q}(h,\eta,\zeta)$ in each ζ_j. So

$$\widehat{Pu}(\eta) = \sum_{\xi \in \Gamma} \hat{p}(h, \eta-\xi, h\xi)\ \hat{u}(\xi)$$

where $\hat{p}(h,\eta,\zeta)$ is given by (3.4). Define

$$(3.5) \qquad p_K^m(h,\eta) = \sup_{|\beta|\le K} \sup_{\zeta} \left| \partial_\zeta^\beta \hat{p}(h,\eta,\zeta)\, h^{|\beta|} \Lambda(h,\zeta)^{-m+|\beta|} \right| .$$

For $|\beta| \le K$, $\partial_\zeta^\beta \hat{p}(h,\eta,\zeta)\, h^{|\beta|} \Lambda(h,\zeta)^{-m_1-m_2+|\beta|}$ can be written

as a sum of terms of the form

$$\sum_{\theta\in\Gamma} \left(\partial_\zeta^\gamma \hat{q}(h,\eta-\theta,\zeta+h\theta) h^{|\gamma|} \Lambda(h,\zeta)^{-m_1+|\gamma|}\right)\left(\partial_\zeta^{\beta-\gamma} \hat{r}(h,\theta,\zeta)\, h^{|\beta-\gamma|} \Lambda(h,\zeta)^{-m_2+|\beta-\gamma|}\right)$$

for $\gamma\le\beta$, each of which is bounded by

$$\sum_{\theta\in\Gamma} \left(\sqrt{2}\ \lambda(\theta)\right)^\mu q_K^{m_1}(h,\eta-\theta)\ r_K^{m_2}(h,\theta)$$

by Lemma 3.1, where $\mu = \sup_{|\gamma|\le K} \left|-m_1 + |\gamma|\right|$. So

$$\begin{aligned}
\sum_{\eta\in\Gamma} \lambda(\eta)^J p_K^{m_1+m_2}(h,\eta) &\le C \sum_{\eta,\theta\in\Gamma} \lambda(\eta)^J \lambda(\theta)^\mu q_K^{m_1}(h,\eta-\theta)\, r_K^{m_2}(h,\theta) \\
&\le C_1 \sum_{\eta,\theta\in\Gamma} \lambda(\eta+\theta)^J \lambda(\theta)^\mu q_K^{m_1}(h,\eta)\, r_K^{m_2}(h,\theta) \\
&\le C_2 \sum_{\eta\in\Gamma} \lambda(\eta)^J q_K^{m_1}(h,\eta) \sum_{\theta\in\Gamma} \lambda(\theta)^{J+\mu} r_K^{m_2}(h,\theta)
\end{aligned}$$

which is bounded by Lemma 3.2 since $q \in S^{m_1}$ and $r \in S^{m_2}$.

So by Lemma 3.2, $p \in S^{m_1+m_2}$.

Let $p_0(h,x,\zeta) = q(h,x,\zeta)\ r(h,x,\zeta)$. Then for $x \in G$,

$$p_0(h,x,\zeta) = \sum_{\eta,\theta \in \Gamma} e^{ix\cdot(\eta+\theta)}\ \hat{q}(h,\eta,\zeta)\ \hat{r}(h,\theta,\zeta)$$

$$= \sum_{\eta,\theta \in \Gamma} e^{ix\cdot\eta}\ \hat{q}(h,\eta-\theta,\zeta)\ \hat{r}(h,\theta,\zeta)$$

by the $(2N+1)$-periodicity of $\hat{q}(h,\eta,\zeta)$ in each η_j. So

$$\hat{p}_0(h,\eta,\zeta) = \sum_{\theta \in \Gamma} \hat{q}(h,\eta-\theta,\zeta)\ \hat{r}(h,\theta,\zeta) \tag{3.6}$$

$$\hat{p}_1(h,\eta,\zeta) = \hat{p}(h,\eta,\zeta) - \hat{p}_0(h,\eta,\zeta)$$

$$= \sum_{\theta \in \Gamma} \left(\hat{q}(h,\eta-\theta,\zeta+h\theta) - \hat{q}(h,\eta-\theta,\zeta)\right)\ \hat{r}(h,\theta,\zeta)\ .$$

For $|\beta| \le K$, $\partial_\zeta^\beta\ \hat{p}_1(h,\eta,\zeta)\ h^{|\beta|}\ \Lambda(h,\zeta)^{-m_1-m_2+1+|\beta|}$ can be written as a sum of terms of the form

$$\sum_{\theta \in \Gamma} \left(\partial_\zeta^\gamma \left(v(\zeta+h\theta) - v(\zeta)\right) h^{|\gamma|}\ \Lambda(h,\zeta)^{-m_1+1+|\gamma|}\right)\left(\partial_\zeta^{\beta-\gamma}\ \hat{r}(h,\theta,\zeta)\ h^{|\beta-\gamma|}\ \Lambda(h,\zeta)^{-m_2+|\beta-\gamma|}\right)$$

for any $\gamma \le \beta$ where $v(\zeta) = \hat{q}(h,\eta-\theta,\zeta)$. For fixed $\eta,\theta \in \Gamma$, and ζ,

$$\left|\partial_\zeta^\gamma \left(v(\zeta+h\theta) - v(\zeta)\right)\ h^{|\gamma|}\ \Lambda(h,\zeta)^{-m_1+1+|\gamma|}\right|$$

$$= \left| \int_0^1 \sum_{j=1}^{d} \partial_\zeta^{\gamma+e_j} v(\zeta+th\theta)\, h^{|\gamma|+1} \Lambda(h,\zeta)^{-m_1+1+|\gamma|} \theta_j \, dt \right|$$

$$\leq C\, q_{K+1}^{m_1}(h,\eta-\theta)\, |\theta| \int_0^1 \lambda(t\theta)^{\mu}\, dt$$

by Lemma 3.1, where $\mu = \sup_{|\gamma| \leq K} \left| -m_1+1+|\gamma| \right|$. But for $\theta \in \Gamma$,

$|\theta| \leq \frac{\pi}{2}\lambda(\theta)$ and $\lambda(t\theta) \leq \lambda(\theta)$ for $0 \leq t \leq 1$. So

$$\left| \partial_\zeta^{\gamma} v(\zeta+h\theta) - v(\zeta) \quad h^{|\gamma|} \Lambda(h,\zeta)^{-m_1+1+|\gamma|} \right| \leq C\, \lambda(\theta)^{\mu+1} q_{K+1}^{m_1}(h,\eta-\theta)\ .$$

This implies

$$\sum_{\eta \in \Gamma} \lambda(\eta)^{J}\, p_{1\,K}^{m_1+m_2-1}(h,\eta) \leq C \sum_{\eta,\theta \in \Gamma} \lambda(\eta)^{J} \lambda(\theta)^{\mu+1} q_{K+1}^{m_1}(h,\eta-\theta)\, r_K^{m_2}(h,\theta)$$

$$\leq C_1 \Big(\sum_{\eta \in \Gamma} \lambda(\eta)^{J} q_{K+1}^{m_1}(h,\eta) \Big) \Big(\sum_{\theta \in \Gamma} \lambda(\theta)^{J+\mu+1} r_K^{m_2}(h,\theta) \Big)$$

which is bounded by Lemma 3.2 since $q \in S^{m_1}$ and $r \in S^{m_2}$. So by Lemma 3.2, $p_1 \in S^{m_1+m_2-1}$. This proves Theorem 3.2.

Theorem 3.3 If $q(h,x,\zeta) \in S^{m_1}$, $r(h,x,\zeta) \in S^{m_2}$, and the symbols $q(h,x,\zeta)$ and $r(h,x,\zeta)$ commute, then $QR-RQ \in PS^{m_1+m_2-1}$.

Proof $\sigma_{QR} - \sigma_{RQ} = (\sigma_{QR} - qr) - (\sigma_{RQ} - rq) \in S^{m_1+m_2-1}$ by Theorem 3.2.

The adjoint of a linear operator A on $\ell^2(G)^n$ is the linear operator A^* for which

$$(Au,v)_G = (u,A^*v)_G$$

for all $u, v \in \ell^2(G)^n$. The adjoint of a pseudo-difference operator P is again a pseudo-difference operator whose symbol is, to highest order, the conjugate transpose p^* of the symbol p of the original operator. Note that the $*$ on a symbol denotes a matrix adjoint; on an operator it denotes an operator adjoint.

<u>Theorem 3.4</u> If $p(h,x,\zeta) \in S^m$, then P^* is a pseudo-difference operator with symbol $q(h,x,\zeta) \in S^m$ given by

$$\hat{q}(h,\eta,\zeta) = \hat{p}(h,-\eta,\zeta+h\eta)^*. \tag{3.7}$$

Moreover, if $r(h,x,\zeta) = p(h,x,\zeta)^*$, then $r(h,x,\zeta) \in S^m$ and

$$q(h,x,\zeta) - r(h,x,\zeta) \in S^{m-1}.$$

<u>Proof</u>

$$(u, P^*v)_G = \sum_{\xi \in \Gamma} \widehat{Pu}(\xi)^* \hat{v}(\xi)$$

$$= \sum_{\eta \in \Gamma} \hat{u}(\eta)^* \sum_{\xi \in \Gamma} \hat{p}(h,\xi-\eta,h\eta)^* \hat{v}(\xi).$$

So

$$\widehat{P^*v}(\eta) = \sum_{\xi \in \Gamma} \hat{p}(h,\xi-\eta,h\eta)^* \hat{v}(\xi)$$

$$= \sum_{\xi \in \Gamma} \hat{q}(h,\eta-\xi,h\xi)\ \hat{v}(\xi)$$

where $\hat{q}(h,\eta,\zeta)$ is given by (3.7).

Let $\mu = \sup_{|\beta| \leq K} |-m+|\beta||$. Then by Lemma 3.1, using the notation of (3.5),

$$q_K^m(h,\eta) = \sup_{|\beta| \leq K} \sup_{\zeta} \left| \partial_\zeta^\beta\ \hat{p}(h,-\eta,\zeta+h\eta)^*\ h^{|\beta|}\ \Lambda(h,\zeta)^{-m+|\beta|} \right|$$

$$\leq 2^{\mu/2}\ \lambda(-\eta)^\mu\ p_K^m(h,-\eta)\ .$$

So $\displaystyle\sum_{\eta \in \Gamma} \lambda(\eta)^J\ q_K^m(h,\eta) \leq 2^{\mu/2} \sum_{\eta \in \Gamma} \lambda(\eta)^{J+\mu}\ p_K^m(h,\eta)$

which is bounded by Lemma 3.2 since $p \in S^m$. Hence $q \in S^m$.

Clearly $r \in S^m$ and

$$\hat{q}(h,\eta,\zeta) - \hat{r}(h,\eta,\zeta) = \hat{p}(h,-\eta,\zeta+ h\eta)^* - \hat{p}(h,-\eta,\zeta)^*.$$

Let $v(\zeta) = \hat{p}(h,-\eta,\zeta)^*$. As in the proof of Theorem 3.2, we can show that

$$\left| \partial_\zeta^\beta \Big(v(\zeta+h\eta) - v(\zeta)\Big)\, h^{|\beta|}\ \Lambda(h,\zeta)^{-m+1+|\beta|} \right| \leq C\ \lambda(\eta)^{\mu+1}\ p_{K+1}^m(h,-\eta)$$

where $\mu = \sup_{|\beta| \leq K} |-m+1+|\beta||$. So

$$\sum_{\eta \in \Gamma} \lambda(\eta)^J\ (q-r)_K^{m-1}\ (h,\eta) \leq C \sum_{\eta \in \Gamma} \lambda(\eta)^{J+\mu+1}\ p_{K+1}^m\ (h,\eta)$$

which is bounded by Lemma 3.2. Hence $q-r \in S^{m-1}$.

We complete this section with a proof of Gårding's Inequality. We need first an interpolation inequality.

Lemma 3.3 Let $s_1 < s_2 < s_3$ be real numbers. Given $\varepsilon>0$, there is a constant C_ε depending on ε, s_1, s_2, s_3 such that

$$\|u\|_{s_2}^2 \leq \varepsilon \|u\|_{s_3}^2 + C_\varepsilon \|u\|_{s_1}^2 . \tag{3.8}$$

Proof For $\xi \in \Gamma$,

$$\frac{4}{\pi^2}(1+|\xi|^2) \leq \lambda(\xi)^2 \leq (1+|\xi|^2) .$$

Choose $M>0$ so large that $\left(\frac{4}{\pi^2}(1+M^2)\right)^{s_2-s_3} \leq \varepsilon$.

Let $\Gamma_1 = \{\xi \in \Gamma: |\xi| \leq M\}$, $\Gamma_2 = \{\xi \in \Gamma: |\xi| > M\}$,

and $C_\varepsilon = (1+M^2)^{(s_2-s_1)}$. Then

$$\|u\|_{s_2}^2 = \sum_{\xi \in \Gamma_1} \lambda(\xi)^{2s_2} |\hat{u}(\xi)|^2 + \sum_{\xi \in \Gamma_2} \lambda(\xi)^{2s_2} |\hat{u}(\xi)|^2$$

$$\leq (1+M^2)^{s_2-s_1} \sum_{\xi \in \Gamma_1} \lambda(\xi)^{2s_1} |\hat{u}(\xi)|^2$$

$$+ \left(\frac{4}{\pi^2}(1+M^2)\right)^{s_2-s_3} \sum_{\xi \in \Gamma_2} \lambda(\xi)^{2s_3} |\hat{u}(\xi)|^2$$

$$\leq C_\varepsilon \|u\|_{s_1}^2 + \varepsilon \|u\|_{s_3}^2 .$$

If p is an nxn matrix, define $\operatorname{Re} p = (p + p^*)/2$. If $pz \cdot z \geq C|z|^2$ for all $z \in \mathbb{C}^n$ (where $w \cdot z = \sum \overline{w}_j z_j$) we write $p \geq CI$ or just $p \geq C$. If P is a linear operator on $\ell^2(G)^n$, define $\operatorname{Re} P = (P + P^*)/2$.

<u>Theorem 3.5</u> (Gårding's Inequality) If $p(h,x,\zeta) \in S^0$ is an nxn matrix symbol and $\operatorname{Re} p(h,x,\zeta) \geq C_0 > 0$, then for any $\varepsilon>0$ and $s<0$ there is a constant $C_{s,\varepsilon}$ such that

$$\operatorname{Re}(Pu,u)_G \geq (C_0-\varepsilon)\|u\|_0^2 - C_{s,\varepsilon}\|u\|_s^2 . \tag{3.9}$$

<u>Proof</u> We first show that if $r(h,x,\zeta) \in S^0$ is a hermitian matrix symbol and $r(h,x,\zeta) \geq C_1 > 0$, then the matrix square root $b(h,x,\zeta)$ of $r(h,x,\zeta)$ (the unique hermitian matrix with positive eigenvalues for which $b^2 = r$) is also in S^0 and

$$\operatorname{Re} R - B^*B \in PS^{-1} . \tag{3.10}$$

Since r is hermitian and in S^0, there is a positive constant C_2 such that all the eigenvalues of $r(h,x,\zeta)$ for any h,x,ζ are in the real interval $[C_1, C_2]$. Let γ be a smooth curve in the right half plane in $\mathbb{C}$ surrounding $[C_1,C_2]$. Then

$$b(h,x,\zeta) = \frac{1}{2\pi i}\int_\gamma \sqrt{z}\,\big(zI - r(h,x,\zeta)\big)^{-1}\,dz \tag{3.11}$$

where $\sqrt{z}$ is the branch of the square root which is positive on the positive real axis; see Chapter 10 of Rudin [1973] for details. For any $z \in \gamma$ and any h, x, ζ, $|(zI-r(h,x,\zeta))^{-1}|$ is bounded by the reciprocal of the distance from γ to $[C_1,C_2]$. If $A(t)$ is a matrix function of a real variable t, then

$$\text{(3.12)} \qquad \frac{d}{dt} A^{-1} = -A^{-1} \frac{dA}{dt} A^{-1} \quad \text{and} \quad \delta A^{-1} = -A(t+h)^{-1} \cdot \delta A \cdot A(t)^{-1}.$$

Differentiating and differencing under the integral sign in equation (3.11), it is easy to show by induction on $|\alpha| + |\beta|$ that $b(h,x,\zeta)$ satisfies Definition 3.1 with $m = 0$. Theorem 3.4 implies $\sigma_{ReR} - r \in S^{-1}$ and $b - \sigma_{B^*} \in S^{-1}$, so $r - \sigma_{B^*} b = (b - \sigma_{B^*}) b \in S^{-1}$. Theorem 3.3 implies $\sigma_{B^*} b - \sigma_{B^*B} \in S^{-1}$, so $\sigma_{ReR} - \sigma_{B^*B} \in S^{-1}$, proving (3.10).

Next we prove the theorem for the case $s = -\frac{1}{2}$. Let $r_\varepsilon = \frac{1}{2}(p+p^*) - (C_0 - \varepsilon)I$. Then $r_\varepsilon \geq \varepsilon$, $r_\varepsilon \in S^0$, and r_ε is hermitian, so $b_\varepsilon = r_\varepsilon^{\frac{1}{2}} \in S^0$ and $\sigma_{ReR_\varepsilon} - \sigma_{B^*B_\varepsilon} \in S^{-1}$. By theorem 3.4, $\sigma_{ReP} - \sigma_{ReR_\varepsilon} - (C_0 - \varepsilon)I \in S^{-1}$, so

$$q_\varepsilon = (C_0-\varepsilon)I + \sigma_{B^*_\varepsilon B_\varepsilon} - \sigma_{ReP} \in S^{-1}.$$

By Theorem 3.1,

$$\begin{aligned}(Q_\varepsilon u,u)_G &= (\Lambda^{\frac{1}{2}}Q_\varepsilon u,\Lambda^{-\frac{1}{2}}u)_G \\ &\leq \|\Lambda^{\frac{1}{2}}Q_\varepsilon u\|_0 \cdot \|\Lambda^{-\frac{1}{2}}u\|_0 \\ &\leq C_\varepsilon \|u\|^2_{-\frac{1}{2}}.\end{aligned}$$

so
$$(C_0-\varepsilon)\ \|u\|_0^2 = ((\mathrm{Re}\ P)u,u)_G - (B_\varepsilon^* B_\varepsilon u,u)_G + (Q_\varepsilon u,u)_G$$

$$= \mathrm{Re}(Pu,u)_G - \|B_\varepsilon u\|_0^2 + (Q_\varepsilon u,u)_G ,$$

$$(3.13) \qquad (C_0-\varepsilon)\ \|u\|_0^2 \le \mathrm{Re}(Pu,u)_G + C_\varepsilon \|u\|_{-\frac{1}{2}}^2$$

which proves (3.9) for $s = -1/2$.

Using (3.13) with ε replaced by $\varepsilon/2$ and Lemma 3.3 with ε replaced by $\frac{\varepsilon}{2C_{\varepsilon/2}}$ (where $C_{\varepsilon/2}$ comes from (3.13)),

$$\left(C_0 - \frac{\varepsilon}{2}\right) \|u\|_0^2 \le \mathrm{Re}(Pu,u)_G + C_{\varepsilon/2}\left(\frac{\varepsilon}{2C_{\varepsilon/2}} \|u\|_0^2 + C' \|u\|_s^2\right)$$

$$(C_0 - \varepsilon)\ \|u\|_0^2 \le \mathrm{Re}(Pu,u)_G + C_{s,\varepsilon} \|u\|_s^2$$

which proves (3.9).

4. Regularity Estimates on a Torus and Proof of Theorem 2.1

In this section we prove the interior regularity estimate Theorem 2.1 and analogous results for regular elliptic pseudo-difference operators defined on the torus T^d, which we now define.

<u>Definition 4.1</u> Let $\sigma,\tau \in \mathbb{Z}^n$. Suppose $\ell_{ij}(h,x,\zeta) \in S^{\sigma_i+\tau_j}$ for $i,j = 1,\cdots,n$ and $\ell(h,x,\zeta)$ is the matrix symbol whose elements are $\ell_{ij}(h,x,\zeta)$. We say that $\ell(h,x,\zeta)$ is regular elliptic of order (σ,τ) if there are positive constants C, ξ_0, h_0 such that

(4.1) $$|\det \ \ell(h,x,\zeta)| \ \geq \ C\Lambda(h,\zeta)^{2p}$$

for $h\xi_0 \leq |\zeta|_\infty \leq \pi$ and $h \leq h_0$ where $2p = \sum_{i=1}^{n} (\sigma_i + \tau_i)$.
If $\sigma_i = 0$ and $\tau_i = m$ for all i, we say that $\ell(h,x,\zeta)$ is regular elliptic of homogeneous order m. We say that a pseudo-difference operator L is regular elliptic of order (σ,τ) or of homogeneous order m if its symbol is. This definition is obviously the pseudo-difference analogue of definitions (2.1) and (2.2).

<u>Lemma 4.1</u> If $p(h,x,\zeta) \in S^m$ is regular elliptic of homogeneous order $m \geq 0$, then for $h\xi_0 \leq |\zeta|_\infty \leq \pi$ and $h \leq h_0$, the matrix $p(h,x,\zeta)$ is invertible and there is a constant C such that

(4.2) $$|p(h,x,\zeta)^{-1}| \leq C\ \Lambda(h,\zeta)^{-m} .$$

<u>Proof</u> Each element of $p(h,x,\zeta)^{-1}$ is $\left(\det p(h,x,\zeta)\right)^{-1}$ times a sum of terms, each of which is a product of $n-1$ elements of $p(h,x,\zeta)$. So

$$|p(h,x,\zeta)^{-1}| \ \leq \ C\ \Lambda(h,\zeta)^{-nm}\left(\Lambda(h,\zeta)^m\right)^{n-1} \ = C\ \Lambda(h,\zeta)^{-m} .$$

We now construct a parametrix q for the symbol p, that is, an inverse for p modulo $S^{-\infty}$ symbols.

<u>Lemma 4.2</u> If $p(h,x,\zeta) \in S^m$ is regular elliptic of homogeneous order $m \geq 0$, then there exists a symbol $q(h,x,\zeta) \in S^{-m}$ such that

(4.3) $$\psi(h,\zeta) \ = \ I \ - \ q(h,x,\zeta)\ p(h,x,\zeta)$$

is independent of x and $\psi(h,\zeta) \in S^{-\infty}$.

Proof Clearly we may assume $h_0 \leq \frac{\pi}{2\xi_0}$ in Definition 4.1. Choose $\phi(t) \in C^{\infty}(\mathbb{R})$ such that $0 \leq \phi(t) \leq 1$, $\phi(t) = 1$ for $|t| < \xi_0$, and $\phi(t) = 0$ for $|t| > 2\xi_0$. Define

$$\psi(h,\zeta) = \phi\left(\frac{\zeta_1}{h}\right) \cdots \phi\left(\frac{\zeta_d}{h}\right) \text{ for } h \leq h_0,\ |\zeta|_\infty \leq \pi$$

and extend $\psi(h,\zeta)$ to be 2π-periodic in each ζ_j. It is easy to show that $\psi(h,\zeta) \in S^{m_1}_{J,K}$ for all $m_1 \in \mathbb{R}$ and all J,K. Let $\chi(h,\zeta) = 1 - \psi(h,\zeta)$, and define

$$q(h,x,\zeta) = \chi(h,\zeta)\ p(h,x,\zeta)^{-1} .$$

By periodicity, we may assume $|\zeta|_\infty \leq \pi$. Since $\chi(h,\zeta) = 0$ for $|\zeta|_\infty \leq h\xi_0$ and $\det p(h,x,\zeta) \neq 0$ for $h\xi_0 \leq |\zeta|_\infty \leq \pi$, $q(h,x,\zeta)$ is well defined, and equation (4.3) holds.

Using Lemma 4.1 and equations (3.12), it is not difficult to show that $q(h,x,\zeta) \in S^{-m}$. An induction on $|\alpha| + |\beta|$ is needed to verify that $q(h,x,\zeta)$ satisfies Definition 3.1.

Theorem 4.1 Suppose $p(h,x,\zeta) \in S^m$ is regular elliptic of homogeneous order $m \geq 0$ and let s and t be any real numbers. Then there is a constant C independent of h such that if $Pu = f$, then

$$\| u \|_{s+m} \leq C\left(\| f \|_s + \| u \|_t\right) . \qquad (4.4)$$

<u>Proof</u> We may assume $h \leq h_0$ since the estimate (4.4) is clearly true for $h_0 \leq h \leq h_{max}$. Let $q(h,x,\zeta)$ and $\chi(h,\zeta)$ be as in Lemma 4.2. By Theorem 3.2, the symbol of the operator QP can be written as

$$\sigma_{QP}(h,x,\zeta) = q(h,x,\zeta)\ p(h,x,\zeta) + r_1(h,x,\zeta)$$

where $r_1(h,x,\zeta) \in S^{-1}$. Since $qp = \chi = 1 - \psi$,

$$\sigma_{QP}(h,x,\zeta) = I + r(h,x,\zeta)$$

where $r = r_1 - \psi$ is also in S^{-1}. If $Pu = f$, then $QPu = Qf$, so $u = Qf - Ru$. We may assume that $s+m-t$ is a positive integer; lowering the value of t if necessary only makes equation (4.4) stronger. If $t \leq j \leq s+m-1$, then

$$\begin{aligned} \| u \|_{j+1} &\leq \| Qf \|_{j+1} + \| Ru \|_{j+1} \\ &\leq \| Qf \|_{s+m} + \| Ru \|_{j+1} \\ &\leq C\left(\| f \|_s + \| u \|_j\right). \end{aligned}$$

This last inequality is a result of Theorem 3.1. Applying this inequality for $j = t, t+1, \cdots, s+m-1$, we obtain

$$\| u \|_{s+m} \leq C\left(\| f \|_s + \| u \|_t\right).$$

We now apply Theorem 4.1 to obtain an estimate similar to equation (4.4) for regular elliptic symbols of order (σ,τ).

<u>Lemma 4.3</u> If $p(h,x,\zeta) \in S^m$ is regular elliptic of homogeneous order m and $r(h,x,\zeta) \in S^{m_1}$ with $m_1 < m$, then $p(h,x,\zeta) + r(h,x,\zeta)$ is regular elliptic of homogeneous order m.

<u>Proof</u> Expanding out the products in $\det\big(p(h,x,\zeta) + r(h,x,\zeta)\big)$, we get $\det p(h,x,\zeta)$ plus a sum of terms, each of which is a product of n factors of elements of p or r with at least one factor coming from r; each term in this latter sum is bounded by $\Lambda(h,\zeta)^{(n-1)m + m_1}$. Thus

$$|\det\,(p(h,x,\zeta) + r(h,x,\zeta))| \geq C\Lambda(h,\zeta)^{nm} - C_1\,\Lambda(h,\zeta)^{(n-1)m+m_1}$$

for $h\xi_0 \leq |\zeta|_\infty \leq \pi$. Since $\Lambda(h,\zeta) \geq \frac{2}{\pi h}\ |\zeta|_\infty$ for $|\zeta|_\infty \leq \pi$ and $nm > (n-1)m + m_1$, $C\Lambda(h,\zeta)^{nm}$ dominates $C_1\Lambda(h,\zeta)^{(n-1)m+m_1}$ if we increase ξ_0 sufficiently. The lemma follows.

<u>Definition 4.2</u> If $\sigma \in \mathbb{Z}^n$, define the matrix symbol

$$\Lambda^\sigma(h,\zeta) = \operatorname{diag}\left(\Lambda(h,\zeta)^{\sigma_1}, \cdots \Lambda(h,\zeta)^{\sigma_n}\right),$$

the diagonal matrix with diagonal entries $\Lambda(h,\zeta)^{\sigma_1}, \cdots, \Lambda(h,\zeta)^{\sigma_n}$.

Let Λ^σ denote the operator with symbol $\Lambda^\sigma(h,\zeta)$. If $s \in \mathbb{R}$, define

$$\| u \|^2_{s+\sigma} = \| \Lambda^{\sigma} u \|^2_s = \sum_{i=1}^{n} \| u_i \|^2_{s+\sigma_i} .$$

<u>Theorem 4.2</u> If $\ell(h,x,\zeta)$ is regular elliptic of order (σ,τ) and s and t are real numbers, then there is a constant C independent of h such that if $Lu = f$, then

$$\|u\|_{s+\tau} \leq C\left(\|f\|_{s-\sigma} + \|u\|_t\right) . \tag{4.5}$$

<u>Proof</u> Let $p(h,x,\zeta) = \Lambda^{-\sigma}(h,\zeta)\, \ell(h,x,\zeta)\, \Lambda^{-\tau}(h,\zeta)$. Then

$$p_{ij}(h,x,\zeta) = \ell_{ij}(h,x,\zeta)\Lambda(h,\zeta)^{-\sigma_i - \tau_j} \in S^0.$$

So $p(h,x,\zeta)$ is regular elliptic of homogeneous order 0. Note that $P_{ij} = L_{ij}\,\Lambda^{-\sigma_i - \tau_j}$. Define $\tilde{P} = \Lambda^{-\sigma} L \Lambda^{-\tau}$. Then $\tilde{P}_{ij} = \Lambda^{-\sigma_i} L_{ij} \Lambda^{-\tau_j}$.

By Theorem 3.3, the symbol of

$$\tilde{P}_{ij} - P_{ij} = \left(\Lambda^{-\sigma_i} L_{ij} - L_{ij} \Lambda^{-\sigma_i}\right) \Lambda^{-\tau_j}$$

is in S^{-1}. By Lemma 4.3, $\tilde{P}$ is regular elliptic of homogeneous order 0. Suppose $Lu = f$. Then

$$\tilde{P}\,\Lambda^{\tau} u = \Lambda^{-\sigma} L\, u = \Lambda^{-\sigma} f$$

so by Theorem 4.1,

$$\| \Lambda^{\tau} u\|_s \leq C\left(\|\Lambda^{-\sigma} f\|_s + \| \Lambda^{\tau} u\|_{t_1}\right)$$

with t_1 arbitrarily small. Hence

$$\|u\|_{s+\tau} \leq C\left(\|f\|_{s-\sigma} + \|u\|_t\right).$$

We remark that equation (4.2) is the only property of regular elliptic symbols of homogeneous order m (beyond the fact that they are symbols) which is used in the proof of Theorem 4.1. We could have used other criteria in Definition 4.1 for these symbols which lead to (4.2), for example,

$$|p(h,x,\zeta)z \cdot z| \leq C\Lambda(h,\zeta)^m |z|^2 \qquad \text{for all} \quad z \in \mathbb{C}^n$$

for $h\xi_0 \leq |\zeta|_\infty \leq \pi$ and $h \leq h_0$.

We now present the proof of Theorem 2.1.

<u>Proof of Theorem 2.1</u> Suppose the regular elliptic system of difference equations (2.10) is defined in Ω. Let M be the union of all the sets M which occur when we write the difference operators L_{ij} in the form of equation (2.6). If $\mathcal{O}$ is an open subset of Ω and $\mathcal{O}_{h,M}$ is defined as in equation (2.16), then for a given h and $x \in \mathcal{O}_{h,M}$, $Lu(x)$ depends only on the values of u in $\mathcal{O}$.

Let Ω_1 be a subdomain with compact closure $\overline{\Omega}_1 \subset \Omega$. Without loss of generality, we may assume that there is an open cube Q such that $\overline{\Omega}_1 \subset Q \subset \overline{Q} \subset \Omega$; any Ω_1 is the union of a finite number of subdomains of Ω which do have this property, and the estimate (2.17) for Ω_1 can be obtained from the estimates (2.17) for these subdomains.

For any fixed $h_1 > 0$, estimate (2.17) is clearly true if we restrict h to the interval $[h_1, h_{max}]$. We will choose h_1 to satisfy a finite number of conditions. Then we only have to prove (2.17) for $0 < h \le h_1$.

We may assume Q is $(0,2\pi)^d$. Although Q is a subset of $\Omega \subset \mathbb{R}^d$, we may also identify Q with the open subset $(0,2\pi)^d$ of T^d so that we may use the theory of pseudo-difference operators which we have developed. We may also assume that for $h \le h_1$, the grids $G(h,\Omega) \cap Q$ considered are all grids on the torus of the kind dealt with in section 3, i.e., $h = 2\pi/(2N+1)$. Although the theorem is stated for a continuum of values of h, we will consider only a discrete set of values of h tending to zero; clearly the details necessary to extend our arguments to all h in $0 < h \le h_1$ can be worked out.

Let $r_m = \max \tau_j + k + 1$. For $r = 1,2,\cdots,r_m$, let ϕ_r be C^∞ functions defined on Q with compact support K_r such that: $0 \le \phi_r \le 1$; $\phi_1 \equiv 1$ on a neighborhood of $\overline{\Omega}_1$; for $1 \le r \le r_m - 1$, $\phi_{r+1} \equiv 1$ on a neighborhood of K_r; and $K_{r_m} \subset Q$. Let $\mathcal{O}^r$ be the interior of the set where $\phi_r \equiv 1$, and set $\mathcal{O}^{r_m+1} \equiv Q$. Then $K_r \subset \mathcal{O}^{r+1}$ for $1 \le r \le r_m$.

Choose $h_1 > 0$ so small that for $0 < h \le h_1$, the following holds: for each $x \in \mathbb{R}^d$ for which $R(h,x,k+\tau) \cap \overline{\Omega}_1 \ne \emptyset$, $R(h,x,k+\tau) \subset \mathcal{O}^1$; $K_r \subset \mathcal{O}^{r+1}_{h,M}$ for $1 \le r \le r_m$; for each $x \in \mathbb{R}^d$ for which $R(h,x,k-\sigma) \cap K_{r_m} \ne \emptyset$, $R(h,x,k-\sigma) \subset Q$; and $\overline{Q} \subset \Omega_{h,M}$. For the rest of the proof assume $h \le h_1$. Then

$$\|u\|_{h,k+\tau,\Omega_1} \le \|\phi_1 u\|_{h,k+\tau,\,T^d} \le C\,\|\phi_1 u\|_{k+\tau} \,. \tag{4.6}$$

For $1 \le r \le r_m$,

(4.7) $$\| \phi_r f \|_{k-\sigma} \le C\|\phi_r f\|_{h,k-\sigma,T^d} \le C\|f\|_{h,k-\sigma,Q} \cdot$$

Let ψ: $\mathbb{R}^d \to Q$ be a C^∞ function, 2π-periodic in each x_j, which is the identity on a neighborhood of K_{r_m}. Define $\tilde{L}$ on T^d to be the operator with symbol

$$\tilde{\ell}(h,x,\zeta) = \ell(h,\psi(x),\zeta).$$

Clearly $\tilde{L}$ is a regular elliptic difference operator on T^d of order (σ,τ), and for $1 \le r \le r_m -1$,

(4.8) $$\phi_r L u = \phi_r L(\phi_{r+1}u) = \phi_r \tilde{L}(\phi_{r+1}u) .$$

Let s be an integer, $1 - \max \tau_j \le s \le k$, and let $r = k+1 - s$. Then $1 \le r \le r_m -1$. Applying Theorem 4.2 to $\tilde{L}$ and $\phi_r u$,

(4.9) $$\| \phi_r u\|_{s+\tau} \le C(\| \tilde{L}(\phi_r u) \|_{s-\sigma} + \| \phi_r u\|_0).$$

Now

$$\tilde{L}(\phi_r u) = \tilde{L}(\phi_r \phi_{r+1} u)$$

$$= (\tilde{L} \Phi_r - \Phi_r \tilde{L})(\phi_{r+1}u) + \phi_r \tilde{L}(\phi_{r+1}u)$$

$$= (\tilde{L} \Phi_r - \Phi_r \tilde{L})(\phi_{r+1}u) + \phi_r Lu$$

where Φ_r is the operator with symbol $\phi_r(x) \in S^0$. By Theorem 3.3,

$(\tilde{L}\,\Phi_r - \Phi_r\,\tilde{L})_{ij} \in PS^{\sigma_i + \tau_j - 1}$, so

$$\|\tilde{L}(\phi_r u)\|_{s-\sigma} \leq C\|\phi_{r+1}u\|_{s-1+\tau} + \|\phi_r f\|_{s-\sigma}\,. \tag{4.10}$$

Combining (4.9), (4.10), (4.7), and $s \leq k$,

$$\|\phi_r u\|_{s+\tau} \leq C\left(\|\phi_{r+1}u\|_{s-1+\tau} + \|\phi_r f\|_{k-\sigma} + \|\phi_r u\|_0\right)$$

$$\|\phi_r u\|_{s+\tau} \leq C\left(\|\phi_{r+1}u\|_{s-1+\tau} + \|f\|_{h,k-\sigma,Q} + \|u\|_{h,0,Q}\right)\,. \tag{4.11}$$

Applying (4.11) successively, starting with $s=k$, $r=1$, ending with $s = 1-\max\tau_j$, $r = r_m-1$, and noting that

$$\|\phi_{r_m}u\|_{-(\max\tau_j)+\tau} \leq \|\phi_{r_m}u\|_0 \leq C\|u\|_{h,0,Q}$$

we obtain

$$\|\phi_1 u\|_{k+\tau} \leq C\left(\|f\|_{h,k-\sigma,Q} + \|u\|_{h,0,Q}\right)\,. \tag{4.12}$$

The theorem follows from equations (4.6) and (4.12).

5. Discussion and Examples

This section is written to be a guide to the main results of this paper and to illustrate some of the important concepts. We begin with a discussion of the two difference schemes which were presented in Section 1. We then present some regular and non-regular difference schemes for the Stokes equations. These illustrate the interior regularity results of Section 4. We close this section with a short discussion of the factors to be considered in choosing a difference scheme.

Consider again the examples of difference schemes given in Section 1 for the Cauchy-Riemann equations (1.1) on the unit square. For the boundary data take

$$\begin{aligned} u(0,y) &= -y^2 \qquad & u(1,y) &= 1-y^2 \\ v(x,0) &= 0 \qquad & v(x,1) &= 2x. \end{aligned} \tag{5.1}$$

The solution to the partial differential equations with this boundary data is

$$\begin{aligned} u(x,y) &= x^2 - y^2 \\ v(x,y) &= 2xy. \end{aligned}$$

The centered difference scheme (1.2) is not regular since its symbol is

$$\begin{pmatrix} i \sin \theta/h & -i \sin \phi/h \\ i \sin \phi/h & i \sin \theta/h \end{pmatrix}$$

and the determinant of the symbol is

$$-(\sin^2\theta + \sin^2\phi)h^{-2}.$$

Since this determinant vanishes for $\zeta = (\theta,\phi) = (\pi,\pi)$ the scheme cannot satisfy an estimate of the form (2.11) and thus it is non-regular. This scheme requires boundary conditions in addition to (5.1) to determine the solution. For these we use one-sided differences at the boundary, i.e.

$$\frac{u_{1,j}-u_{0,j}}{h} - \frac{(v_{0,j+1}-v_{0,j-1})}{2h} = 0 \qquad j = 1,\cdots,N-1$$

$$\frac{u_{N,j}-u_{N-1,j}}{h} - \frac{(v_{N,j+1}-v_{N,j-1})}{2h} = 0 \qquad j = 1,\cdots,N-1$$

(5.2)

$$\frac{u_{i+1,0}-u_{i-1,0}}{2h} - \frac{v_{i,1}-v_{i,0}}{h} = 0 \qquad i = 1,\cdots N-1$$

$$\frac{u_{i+1,N}-u_{i-1,N}}{2h} - \frac{v_{i,N}-v_{i,N-1}}{h} = 0 \qquad i = 1,\cdots N-1 .$$

Here $h = 1/N$ and N is an even integer.

The solution to the difference equations (1.2) with boundary data (5.1) and (5.2) is

(5.3)
$$\begin{aligned} u_{ij} &= i^2h^2 - j^2h^2 + \varepsilon_i = x_i^2 - y_j^2 + \varepsilon_i \\ v_{ij} &= 2ijh^2 = 2x_iy_j \quad , \end{aligned}$$

where $\varepsilon_i = 0$ for i even and $\varepsilon_i = -h^2$ for i odd. This solution is second order accurate as one would expect from the formal second-order accuracy of the scheme, but note that the solution is not as smooth as one would expect based on the regularity properties of the differential equations. In particular $(\delta_+^x)^3 u_{ij} = \mathcal{O}(h^{-1})$ and thus no estimate of the form (2.17) in Theorem 2.1 can hold for this scheme.

The staggered scheme (1.3) is easily seen to be a regular scheme and thus satisfies interior regularity estimates by Theorem 2.1. In fact, the solution of scheme (1.3) with boundary data (5.1) agrees with the solution of the differential equations at the grid points.

The divided difference with respect to x of u_{ij} from equation (5.3) is only a first-order accurate approximation to the derivative of u, i.e.

$$\delta_+^x u_{ij} = 2\, x_{i+1/2} \pm h = \left.\frac{\partial u}{\partial x}\right|_{x_{i+1/2}} \pm h.$$

This loss of accuracy is a consequence of the non-regularity of the scheme and shows that an estimate such as (2.20) of Theorem 2.2 can not hold for this scheme.

As a further illustration of the theory, we examine two difference approximations to the Stokes equations on a uniform grid. The Stokes equations for two-dimensions are given by equations (2.5). We will consider only homogeneous data, i.e. $f_i \equiv 0$. For the first difference approximation consider using the usual five-point stencil for the Laplacian and central differences for the first derivatives. This can be written as

$$\left(\delta_+^x \delta_-^x + \delta_+^y \delta_-^y\right) u_{ij} + \delta_0^x p_{ij} = 0$$

(5.4)
$$\left(\delta_+^x \delta_-^x + \delta_+^y \delta_-^y\right) v_{ij} + \delta_0^y p_{ij} = 0$$

$$\delta_0^x u_{ij} + \delta_0^y v_{ij} = 0$$

where δ_0^x is the central difference operator in x, $\delta_0^x = \left(\delta_+^x + \delta_-^x\right)/2$, and similarly for δ_0^y.

The symbol for this system of difference equations is

$$\begin{pmatrix} -\Lambda_0^2 & 0 & (i \sin \theta)/h \\ 0 & -\Lambda_0^2 & (i \sin \phi)/h \\ (i \sin \theta)/h & (i \sin \phi)/h & 0 \end{pmatrix}$$

where $\Lambda_0^2 = 4\left(\sin^2(\tfrac{1}{2}\theta) + \sin^2(\tfrac{1}{2}\phi)\right) h^{-2}$, and its determinant is

$$-4(\sin^2(\tfrac{1}{2}\theta) + \sin^2(\tfrac{1}{2}\phi))(\sin^2\theta + \sin^2\phi)h^{-4} .$$

Note that the determinant vanishes at $\zeta = (\theta,\phi) = (0,\pi)$, $(\pi,0)$, and (π,π) in addition to $\zeta = (0,0)$. This implies that inequality (2.11) can not hold and thus this approximation is a non-regular system of difference equations.

To remedy the non-regularity, we replace the central differences by one-sided differences, such as,

$$\left(\delta_+^x \delta_-^x + \delta_+^y \delta_-^y\right) u_{ij} + \delta_-^x p_{ij} = 0$$

(5.5)
$$\left(\delta_+^x \delta_-^x + \delta_+^y \delta_-^y\right) v_{ij} + \delta_-^y p_{ij} = 0$$

$$\delta_+^x u_{ij} + \delta_+^y v_{ij} = 0.$$

The symbol for this system is

$$\begin{pmatrix} -\Lambda_0^2 & 0 & (1 - e^{-i\theta})/h \\ 0 & -\Lambda_0^2 & (1 - e^{-i\phi})/h \\ (e^{i\theta} - 1)/h & (e^{i\phi} - 1)/h & 0 \end{pmatrix}$$

with determinant

$$-\Lambda_0^4 = 16\left(\sin^2(\tfrac{1}{2}\theta) + \sin^2(\tfrac{1}{2}\theta)\right)^2 h^{-4} .$$

Clearly the estimate (2.11) is satisfied.

It follows from Theorem 2.1 that since the scheme (5.5) is regular, it satisfies interior regularity estimates analogous to the regularity estimates for the Stokes equations. In particular,

$$\|u\|_{h,k+1,\Omega_1} + \|v\|_{h,k+1,\Omega_1} + \|p\|_{h,k,\Omega_1}$$

$$\leq C(k,\Omega_1,\Omega)\left(\|u\|_{h,0,\Omega} + \|v\|_{h,0,\Omega} + \|p\|_{h,0,\Omega}\right)$$

for $k > 0$ and domains Ω_1 and Ω satisfying the hypotheses of Theorem 2.1. The system (5.4) does not satisfy such an estimate.

The system (5.5) is formally second-order accurate if one uses a staggered mesh and makes the following assignments,

$$u_{ij} = u(x_i, y_j)$$

$$v_{ij} = v(x_i + \tfrac{1}{2}h,\ y_j + \tfrac{1}{2}h)$$

$$p_{ij} = p(x_i + \tfrac{1}{2}h,\ y_j)\ .$$

By assigning u_{ij}, v_{ij}, and p_{ij} to the location (x_i, y_j) the scheme (5.4) is formally second-order accurate and the scheme (5.5) is formally first-order accurate.

We offer these comments on the use of regular and non-regular schemes. For a difference scheme to be regular and to have more than first order accuracy it frequently happens that the mesh must be a staggered mesh. At least this is true for the examples of systems considered in this paper. For simple linear elliptic systems such as the Cauchy-Riemann equations (1.2) and the Stokes equations (2.5), staggered regular schemes present no serious difficulties to implement. However, for more complicated systems, staggered grids can cause problems. Boundary conditions with staggered grids present difficulties when they involve variables which are not defined on the boundary coordinate line. Also the addition of lower order terms to the systems (1.1) or (2.5) would necessitate the averaging of variables in the regular difference approximation (1.3) and (5.5) if the formal second-order accuracy is to be maintained.

For more complicated equations involving lower order non-linear terms, as in the incompressible Navier-Stokes equations, or for situations in which the elliptic system is only part of a larger system of equations, the advantage of regularity may not be worth the additional complexity entailed by the staggered grid. In such situations a non-regular difference scheme may present the great advantage of being simpler to implement.

As an example of a non-linear system similar to the Cauchy-Riemann equations which is solved using a staggered grid, we refer to Dendy and Wendroff [1979].

For a non-regular scheme, the inability of the determinant to satisfy inequality (2.11) indicates that the high frequency components of the solution do not depend continuously on the data of the continuous problem. The elimination of these spurious, high frequency components requires a careful treatment at the boundaries.

Moreover, as shown in the first example discussed in this section, non-regularity of a scheme can cause the divided differences of the solution to approximate derivatives with less accuracy than that of the overall scheme. For regular schemes this is not so, as shown by Theorem 2.2. This provides a partial answer to the questions posed by Roache [1974, page 342] on the relative accuracy of the staggered meshes as opposed to non-staggered meshes.

As a general rule then, regular difference schemes give smoother (i.e. regular) solutions at the price of greater complexity of implementation when staggered grids are required. Non-regular schemes can be simpler than regular schemes but require greater care to maintain the necessary accuracy and smoothness of the solution.

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